УДК 519.53, 517.987

МЕТРИЧНІ ХАРАКТЕРИСТИКИ НЕСКІНЧЕННО ВИМІРНОГО КООРДИНАТНОГО ПАРАЛЕПІПЕДА

Романов Володимир Олександрович

Центральноукраїнський державний педагогічний університет імені Володимира Винниченка, м. Кропивницький, Україна

Введені поняття середньо геометричного та індукованого індексів нескінченно вимірного координатного паралелепіпеда, а також досліджені властивості цих індексів.
Ключові слова: середньо геометричний індекс, індукований індекс.

METRIC CHARACTERISTICS OF INFINITE-DIMENSIONAL COORDINATE PARALLELEPIPEDS

Romavov Volodymyr Oleksandrovych

Volodymyr Vynnychenko Central Ukrainian State Pedagogical University, Kropyvnytskyi, Ukraine

It are introduced the notions of the geometrical mean and reduced indexes of infinite-dimensional parallelepiped of coordinates and also it are investigated the properties of these indexes.
Keywords: geometric mean index, reduced index.

Introduction. The family of coordinate parallelepipeds is one of significant classes of sets. Therefore it is important to investigate the geometrical mean and reduced indexes of such parallelepipeds.

Statement of the problem.

Coordinate parallelepiped A with length edges a_k is the product $A = \prod_{k=1}^{\infty}[0, a_k]$ and *n-dimensional base of A* is the product $A_n = \prod_{k=1}^{n}[0, a_k]$.

Geometrical mean index of the n-dimensional set is equal to $\sqrt{SR/V}$, where V is the volume, S is the area of surface and R is the maximum radius of balls which are contained in this set [1]. It is known that value of such index belongs to $(1,\sqrt{n}]$.

Let us denote *the geometrical mean index of n-dimensional base of A by $G_n(A)$.*

Definition 1. The number $g_n(A) = G_n(A)/\sqrt{n}$ is called *the reduced index of n-dimensional base of A.*

Definition 2. If for coordinate parallelepiped A the exist the limits

$$G(A) = \lim_n G_n(A)\ , \quad g(A) = \lim_n g_n(A)\ ,$$

then *G(A)* is called *geometrical mean index of A*, *g(A)* is called *reduced index of A* and *A* is called *measurable with respect to these indexes.*

Remark 1. From the inequalities $1 < G_n(A) \le \sqrt{n}$ it follows that g(A) belongs to [0,1] and G(A) belongs to $[1,\infty]$.

Remark 2. If $g(A) \neq 0$, then $G(A) = \infty$.

The *aim* of the paper is investigation of indexes G and g of infinite-dimensional coordinate parallelepipeds .

Results of paper.

Lemma 1. *For n-dimensional base A_n of A it holds the formula*

$$G_n^2(A_n) = (\textstyle\sum_{k=1}^{n} a_k^{-1})\ \min\{a_k,\ k=1,2,\dots,n\}\ .$$

For *proof it* is sufficiently to note that surface of A_n is the union of 2n hyperfaces with common area $2(\sum_{k=1}^{n} a_k^{-1})(\prod_{k=1}^{n} a_k)$.

Theorem 1. *If sequence (a_n) of edges of parallelepiped A has non-zero limit q and m is infimum of (a_n) , then $g(A) = \sqrt{m/q}$.*

Theorem 1 follows from lemma 1, because $\lim_n (\sum_{k=1}^{n} a_k^{-1}) / n = 1/q$.

Corollary 1. If the sequence (a_n) of parallelepiped edges has non-zero limit, then $g(A) \neq 0$, and for every p from (0, 1] the exist parallelepiped A with condition $\lim_n a_n \neq 0$ such that g(A) = p .

Lemma 2 . *Let $q > 1$ and A has edges $a_n = q^{1-n}$, then $G(A) = \sqrt{\frac{q}{q-1}}$.*

Lemma 3. *If A has edges $a_n = n^{-n}$, then $G(A) = 1$ and therefore $g(A) = 0$.*

Lemma 4. *If A has edges $a_n = (\ln(n+1))^{-1}$ then $g(A) = 1$ and therefore $G(A) = \infty$.*

Lemmas 2-4 follow from lemma 1.

Lemma 5. *Let $q > 0$ and A has edges $a_n = n^{-q}$, then $g(A) = \sqrt{1/q+1}$.*

Proof. For such case $g_n^2(A_n)$ is equal to integral sum for the $\int_0^1 x^q \, dx$ and therefore $g_n(A)$ has correspondent limit.

Theorem 2. *For every p from [0, 1] and every L from [1, ∞] there exist parallelepipeds A and B such that sequences of its edges tend to zero and $g(A) = p$, $G(B) = L$.*

Theorem 2 follows from lemmas 2-5.

Theorem 3. *The exist unmeasurable with respect to G parallelepiped B and unmeasurable with respect to g parallelepiped A such that sequences (b_n) and (a_n) of its edges tend to zero.*

Proof. 1) Let $b_{2m-1} = b_{2m} = 2^{1-m}$. Then from lemma 1 it follows that $\lim_m G_{2m-1}(B) \neq \lim_m G_{2m}(B)$.

2) Let $a_1 = 1$, every of 2 next members of sequence (a_n) is 1/2, every of 4 next is 1/4, every of 8 next is 1/8, and go on. Then for the cases $n = 2^m - 1$ and $n = 2^m$ correspondent subsequences of $g_n(A)$ have different limits. Theorem 3 is proved.

Theorem 4. *1) If edges a_n of parallelepiped A tend to zero and for all coordinate rearrangements T it hold equalities $g(TA) = g(A)$, then $g(A) = 0$.*

2) If edges b_n of parallelepiped B tend to zero, $G(B) < \infty$ and for all coordinate rearrangements T it hold equalities $G(TB) = G(B)$, then $G(B) = 1$.

Proof.

1} Because of invariance with respect to rearrangements we can assume that edges a_n decrease in the wide sense.

The exists increasing sequence $(n(k))$ of natural numbers such that fractions $a_{n(k+1)}/a_{n(k)}$ tend to zero, $n(1) > 1$.

Let T be the coordinate rearrangement such that T(n(1))) = 1, T(n(k+1)) = 1+ n(k) for all k and T(j) = j+1 for the rest. Then n(k)-dimensional bases of A and T(A) have the same collection of edges (although in different orders). Therefore

$$g_{n(k)}(TA) = g_{n(k)}(A).$$

From lemma 1 it follows that $g^2_{1+n(k)}(TA) = g^2_{n(k)}(TA)(a_{n(k+1)}/a_{n(k)}) n(k)(1+n(k))^{-1} + (1+n(k))^{-1} = g^2_{n(k)}(A)(a_{n(k+1)}/a_{n(k)}) n(k)(1 + n(k))^{-1} + (1 + n(k))^{-1}$.

The right part of this equality tends to zero, the left part tends to $g^2(TA)=g^2(A)$, therefore $g(A)=0$.

2) Under analogical assumptions about edges and rearrangement we obtain equality $G^2_{1+n(k)}(TB)=G^2_{n(k)}(B)(b_{n(k+1)}/b_{n(k)}) +1$. Now the right part tends to 1, therefore from condition $G(TB)=G(B)$ it follows that $G(B)=1$. Theorem 4 is proved.

Lemma 6. *Let A has edges $a_{2n-1} = n^{-n}$, $a_{2n} = n^{-1}$ and rearrangement T shifts every a_{2n-1} to right side on the position after a_{2k} with the condition $k^{-1} < n^{-n}$. Then $G(A) =1$, $g(A) =0$, but $G_n(TA)$ does not tend to 1 and $g_n(TA)$ does not tend to zero.*

Lemma 6 follows from lemma 1.

Remark 3. From lemma 6 it follows that theorem 4 gives only necessary conditions of invariance of indexes G and g with respect to all coordinate rearrangements. Sufficient conditions are given by theorem 5.

Theorem 5. *1) If edges a_n of parallelepiped A decrease in the wide sense and tend to zero, then from condition $G(A) =1$ it follows that A is invariant of indexes G and g with respect to all coordinate rearrangements T.*

2) If edges a_n tend to q, where $q\neq 0$, then A also is invariant of G and g, $g^2(A)= \inf\{a_n\}/q > 0$, and therefore $G(A) = \infty$.

Proof. 1) Let us denote $M(n) = \max\{ T(i), i = 1,\dots, n \}$. Then from lemma 1 it follows that $1 < G^2_n(TA) = (\sum_{k=1}^{n} a_{T(k)}^{-1})a_{M(n)} \le G^2_{M(n)}(A)$. The right part tends to 1, therefore $G^2_n(TA)$ also tends to 1. Consequently, $G(TA) =1$, and therefore $g(TA) = 0 = g(A)$.

2) For the case $q \neq 0$ formula for $g(A)$ is given by theorem 1. Right part of this formula is invariant, therefore left part also is invariant. Theorem 5 is proved.

Remark 4. We know (from remark 1) that for every parallelepiped $g(A) \leq 1$. From lemma 4 it follows that in the space c_0 there exist coordinate parallelepipeds A with decreasing edges a_n such that $g(A) = 1$.

Theorem 6. *In spaces l_p (where $p > 0$) there does not exist parallelepiped A with* <u>*decreasing*</u> *edges a_n and such that $g(A) = 1$.*

Proof. Let us suppose that p is the fixed positive number, parallelepiped A has decreasing edges a_n and $g(A) = 1$.

From evident inequality $\frac{1}{2n}\left(\sum_{k=n+1}^{2n} a_k^{-1}\right) a_{2n} \leq 1/2$ and from lemma 1 it follows that $g_{2n}^2(A) - \frac{1}{2} \leq g_{2n}^2(A) - \frac{1}{2n}\left(\sum_{k=n+1}^{2n} a_k^{-1}\right) a_{2n} =$
$\frac{1}{2n}\left(\sum_{k=1}^{n} a_k^{-1}\right) a_{2n} \leq \frac{1}{2} a_{2n} / a_n$, therefore

$$(a_{2n} / a_n)^p \geq (2 g_{2n}^2(A) - 1)^p .$$

The right part of last inequality tends to 1, therefore there exists number $n(0)$ such that $a^p_{2n} / a^p_n > \frac{1}{2}$ for all $n \geq n(0)$.

Then for the numbers $j = 2^k n(0)$ it hold inequalities

$$a^p_{2j} \geq \frac{1}{2} a^p_j \geq 2^{-k-1} a^p_{n(0)} .$$

Then for the partial sums of the series

$$\sum_{m=1}^{\infty} a_m^p$$

are true the inequalities

$$s_{2j} - s_j \geq j\, 2^{-k-1} a^p_{n(0)} = \frac{1}{2} n(0)\, a^p_{n(0)} = \text{const.}$$

Therefore this series is divergent. Theorem 6 is proved.

Список використаних джерел

1. Романов В.А. (2019) Топологические свойства геометрических функционалов и многоугольники с экстремальным значением среднегеометрического индекса. *Наукові записки* Центрально-українського державного педагогічного університету. *Серія математичних наук. 75, 52 – 60.*