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# SOME COMBINATORIAL IDENTITIES THAT IS RELATED TO THE TREES-FUNCTION

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We define the rational functions with coefficients that are generalizations of numbers of Euler the second order. The recurrent relations are got for such coefficients. Using properties of the tree function, we find a few combinatorial identities.

Визначаються раціональні функції, коефіцієнти яких є узагальнення-ми чисел Ейлера другого порядку. Отримано рекурентні співвідношення для таких коефіцієнтів і, використовуючи властивості функції дерев, знайдено декілька комбінаторних тотожностей..

## §1. Sequence $u_{m,s}(t)$

First let us define the sequence of functions $u_{0,s}(t)=\dfrac{1}{s(1-t)^s}$ if $s\neq 0$,

$u_{0,0}(t)=\log\dfrac{1}{1-t}$, $u_{m+1,s}(t)=\dfrac{t}{1-t}u'_{m,s}(t),\ m=0,1,2,\cdots.$

From this it follows that

$u_{1,s}(t)=t/(1-t)^{s+2},\ u_{2,s}(t)=t(1+(s+1)t)/(1-t)^{s+4}$,

$u_{3,s}(t)=t(1+(3s+5)t+(s+1)(s+2)t^2)/(1-t)^{s+6}$,

$u_{4,s}(t)=t(1+(7s+15)t+(6s^2+26s+26)t^2+(s+1)(s+2)(s+3)t^3)/(1-t)^{s+8}$,

$u_{5,s}(t)=t(1+(15s+37)t+(25s^2+135s+168)t^2+(10s^3+80s^2+200s+154)t^3+$
$(s+1)(s+2)(s+3)(s+4)t^4)/(1-t)^{s+10}$,

$$u_{m,s}(t)=t\sum_{k=0}^{m-1}a_{m,k}(s)t^k/(1-t)^{s+2m}. \qquad (1)$$

**Lemma 1.** *The functions* $a_{m,k}(s)$ *satisfy the following the recurrence relation:*

$$a_{m,k}(s)=(k+1)a_{m-1,k}(s)+(2m+s-k-2)a_{m-1,k-1}(s), a_{m,m}(s)=0, a_{m,-1}(s):=0. \qquad (2)$$

*Proof.* Using (1), we get

$$u_{m-1,s}(t)=t\sum_{k=0}^{m-2}a_{m-1,k}(s)t^k/(1-t)^{s+2m-2},$$

which imples that

$$u'_{m-1,s}(t)=\left(\sum_{k=0}^{m-1}(k+1)a_{m-1,k}(s)t^k+\sum_{k=0}^{m-1}(2m+s-k-2)a_{m-1,k}(s)t^k\right)/(1-t)^{s+2m-1}$$

Taking into account of the functions $u_{m,s}(t)$, we have

$$\sum_{k=0}^{m-1} a_{m,k}(s)t^{k+1}/(1-t)^{s+2m} = \left(\sum_{k=0}^{m-1}(k+1)a_{m-1,k}(s)+(2m+s-k-2)a_{m-1,k}(s))t^{k+1}\right)/(1-t)^{s+2m}.$$

Comparing the $t^{k+1}$ terms, we get (2).

*Table* $a_{m,k}(s)$

| m/k | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 |
| 2 | 1 | s+1 | 0 | 0 | 0 |
| 3 | 1 | 3s+5 | (s+1)(s+2) | 0 | 0 |
| 4 | 1 | 7s+15 | $6s^2+26s+26$ | (s+1)(s+2)(s+3) | 0 |
| 5 | 1 | 15s+37 | $25s^2+135s+168$ | $10s^3+80s^2+200s+154$ | (s+1)(s+2)(s+3)(s+4) |

*Table* $a_{m,k}(0)$

| m/k | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 3 | 1 | 5 | 2 | 0 | 0 | 0 | 0 |
| 4 | 1 | 15 | 26 | 6 | 0 | 0 | 0 |
| 5 | 1 | 37 | 168 | 154 | 24 | 0 | 0 |
| 6 | 1 | 83 | 800 | 1792 | 1044 | 120 | 0 |
| 7 | 1 | 177 | 3230 | 14368 | 19556 | 8028 | 720 |

*Table* $a_{m,k}(1)$

| m/k | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 2 | 0 | 0 | 0 | 0 | 0 |
| 3 | 1 | 8 | 6 | 0 | 0 | 0 | 0 |
| 4 | 1 | 22 | 58 | 24 | 0 | 0 | 0 |
| 5 | 1 | 52 | 328 | 444 | 120 | 0 | 0 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6 | 1 | 114 | 1452 | 4400 | 3708 | 720 | 0 |
| 7 | 1 | 240 | 5610 | 32120 | 58140 | 33984 | 5040 |

*Remark.* The numbers of $a_{m,k}(1)$ it the so-called of Euler the second order. ( [2], p.256). These numbers have a combinatorical interpretation. If we form permutations of the multiset $\{1,1,2,2,3,3,\ldots, n,n\}$ which the special property that all numbers between the two occurrences of $m$ are greater than $m$, for $1 \le m \le n$, then is the number of such permutations have $k$ ascents.

**Lemma 2.**

$$\sum_{k=0}^{m-1} a_{m,k}(s) = (s+2)(s+4)\cdots(s+2m-2).$$

*Proof.* Let $P_m(t) := \sum_{i=0}^{m} a_{m+1,i}(s)t^i$ .Then $u_m(t,s) = tP_{m-1}(t)/(1-t)^{2m+s}$,

$u_{m+1}(t,s) = tP_m(t)/(1-t)^{2m+2+s}$, on the other hand $u_{m+1}(t,s) = \frac{t}{1-t}\frac{d}{dt}u_m(t,s)$. So that

$P_m(t)/(1-t)^{2m+1+s} = ((P_{m-1}(t) + tP'_{m-1}(t))(1-t) + (2m+s)tP_{m-1}(t))/(1-t)^{2m+1+s}$. From this $P_m(1) = (2m+s)P_{m-1}(1)$, so that the lemma 2 now follow by induction on the integer $m$ since $P_{m-1}(1) = \sum_{i=0}^{m-1} a_{m,i}(s) = (s+2)(s+4)\cdots(s+2m-2)$.

**Lemma 3.** *The following representation holds:*

$$u_{m,s}(t) = t\sum_{k=0}^{m-1} a_{m,k}(s)t^k/(1-t)^{s+2m} = \sum_{k\ge1} t^k \sum_{i=0}^{m-1} a_{m,i}(s)\binom{k+s+2m-i-2}{2m+s-1}.$$

This relation follow directly from expansion in a series of $(1-t)^{-s-2m}$.

## §2. Basic identity.

Let $T(y)$ is the tree function, id est, this the function, which defined by $T(y) = y\exp(-T(y))$. Then

$$T(y)^k = \sum_{n\ge0} \frac{k}{n!}(n+k)^{n-1} y^{n+k} = \sum_{n\ge1} \frac{n^{n-1}}{n!}\frac{k\cdot k!}{n^k}\binom{n}{k} y^n. \quad (3)$$

$$(1-T(y))^{-s}=\sum_{n\geq 0}t_n(s)\frac{y^n}{n!}, \tag{4}$$

where

$$t_n(s)=\frac{n^{n-1}}{\Gamma(s)}\sum_{k\geq o}\binom{n-1}{y}\frac{\Gamma(s+k+1}{n^k}$$

is tree polynomials ([1], p.339).

The first few cases of the polynomials $t_n(s)$ are

$t_0(s)=1,\ t_1(s)=s,\ t_2(s)=3s+s^2, t_3(s)=17s+9s^2+s^3$.

**Theorem (**Basic identity**).** *The following identity holds:*

$$\sum_{k=1}^{n}\frac{k\cdot k!}{n^k}\binom{n}{k}\sum_{i=0}^{m-1}a_{m,i}(s)\binom{k+s+2m-i-2}{2m+s-1}=\frac{t_n(s)}{s}n^{m-n+1}.$$

*Proof.* From the definition of the tree function we have

$$T'(y)=\frac{T(y)}{y(1-T(y))} \tag{5}$$

If differentiate (3) at $y$ and use (4), we obtain:

$$sT(y)(1-T(y))^{-s-2}=\sum_{n\geq 0}t_n(s)\frac{ny^n}{n!}. \tag{6}$$

If differentiate (4) at $y$ $m$ times and use (5) and (4), we obtain:

$$u_{m,s}(T(y))=\sum_{n\geq 0}\frac{t_n(s)}{s}\frac{n^m y^n}{n!}. \tag{7}$$

If now use of lemma 3, we obtain:

$$u_{m,s}(T(y))=\sum_{k\geq 1}T(y)^k\sum_{i=0}^{m-1}a_{m,i}(s)\binom{k+s+2m-i-2}{2m+s-1}. \tag{8}$$

Further using (3), we obtain:

$$u_{m,s}(T(y)=\sum_{n\geq 1}y^n\sum_{k=1}^{n}\frac{n^{n-1}}{n!}\frac{k\cdot k!}{n^k}\binom{n}{k}\sum_{i=0}^{m-1}a_{m,i}(s)\binom{k+s+2m-i-2}{2m+s-1} \tag{9}$$

Comparing the $y^n$ terms in (7) and (9), we obtain basic identity.

## §3. Corollaries from the basic identity

We use such properties of the tree polynomials ([1]).

**Property 1**. $\lim_{s\to 0}\frac{t_n(s)}{s} = n^{n-1}Q(n)$,

where $Q(n) = \sum_{k=0}^{n-1} n^{-k}(n-1)(n-2)\cdots(n-k)$ is Ramanujan's function.

**Property 2**. $t_n(1) = n^n$.

**Property 3**. $t_n(2) = n^n(Q(n)+1)$.

**Identities.**

**1**. Let $s=0, m=1$.Then

$$\sum_{k=1}^{n}\frac{k^2\cdot k!}{n^k}\binom{n}{k} = nQ(n).$$

**2**. Let $s=0, m=2$. Then

$$\sum_{k=1}^{n}\frac{k^2(2k+1)(k+1)!}{n^k}\binom{n}{k} = 6n^2Q(n).$$

**3**. Let $s=0, m=3$. Then

$$\sum_{k=1}^{n}\frac{k^2(8k^2+11k+1)(k+2)!}{n^k}\binom{n}{k} = 120n^3Q(n).$$

**4**. Let $s=1, m=1$. Then

$$\sum_{k=1}^{n}\frac{k^2\cdot(k+1)!}{n^k}\binom{n}{k} = 2n^2.$$

**5**. Let $s=1, m=2$. Then

$$\sum_{k=1}^{n}\frac{k^2(3k+1)(k+2)!}{n^k}\binom{n}{k} = 24n^3.$$

**6**. Let $s=1, m=3$. Then

$$\sum_{k=1}^{n}\frac{k^3(k+1)(k+3)!}{n^k}\binom{n}{k} = 48n^4.$$

**7**. Let $s=2, m=1$. Then

$$\sum_{k=1}^{n}\frac{k^2(k+2)!}{n^k}\binom{n}{k} = 3n^2(Q(n)+1).$$

**8**. Let $s=2, m=2$. Then

$$\sum_{k=1}^{n}\frac{k^2(4k+1)(k+3)!}{n^k}\binom{n}{k}=60n^3(Q(n)+1).$$

**9**. Let $s=2, m=3$. Then

$$\sum_{k=1}^{n}\frac{k\cdot k!}{n^k}\binom{n}{k}\left[\binom{k+6}{7}+11\binom{k+5}{7}+12\binom{k+4}{7}\right]=\frac{1}{2}n^4(Q(n)+1).$$

**REFERENCES**

[1] Donald E. Knuth and Boris Pittel, A recurrence related to Trees, *Proceedings the American Mathematical Society*, Vol.105, Number 2, 1989, pp.335-349.
[2] Ronald L. Graham, Donald E. Knuth, and Oren Patashnic, *Concrete Mathematics,* Addison-Wesley,1994.