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THE ASYMPTOTIC STABILITY OF THE MAXIMUM

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Let ξ be a random variable that assumes values in $\mathbb{R}^1$ and that has the distribution function $F(x)$. Let ξ_i be independent copies of ξ. Set $z_{n=\max_{1\le i\le n}\xi_i}$.

We say that a sequence (z_n) is relative stable almost surely if there is a numerical sequence (a_n) such that

$$\frac{z_n}{a_n} \to 1 \text{ almost surely} \qquad (1)$$

as $\to\infty$. We also say that (z_n) is stable almost surely if there is numerical sequence (a_n) such that

$$z_n - a_n \to 0 \text{ almost surely} \qquad (2)$$

as $\to\infty$.

Starting with the seminar Gnedenko [1] paper, the (weak) convergence has been studied in the case of degenerate limit laws. The criteria for the asymptotic relations (1) and (2) are also well-known for $\mathbb{R}^1$ (see [2,3]). A survey concerning the convergence of (z_n) in distribution to degenerate laws can be found in [3].

The aim of the current paper is to obtain relations (1) and (2) for the case of infinite dimensional spaces.

The notion of the maximum of two or more random elements can be introduced in the so-called Banach lattices [4]. An important example of Banach lattices is presented by Banach ideal spaces [5].

Let (T,Λ,μ) be a measure space where μ is a σ-finite, σ-additive, and nonnegative measure. By definition, a Banach ideal space B of measurable functions defined in (T,Λ,μ) is a collection of functions such that if $y\in B$ and if $|x(t)\le|y(t)|$ almost surely, then $x\in B$ and $\|x\|\le\|y\|$.

Throughout this paper we consider only the case of separable Banach ideal spaces.

Let B be a Banach ideal space equipped with a norm $\|\cdot\|$ and module $|\cdot|$. Assume that X is a random element that is defined on probability space (Ω, A, P) and that assumed values in the space B. Further let X_i be independent copies of X and

$$Z_n = \max_{1 \le i \le n} X_i.$$

Finally let

$$X = \{X(t), t \in T\}, \qquad \sigma X = \{\sigma(t), t \in T\}$$

$$X(t) = \sigma(t)\tilde{X}(t), \ \ for\ all\ t \in T, \tag{3}$$

and

$$P(\tilde{X}(t) < x) = P(\xi < x) = F(x)$$

for all $t \in T$. The latter two assumptions mean, in particular, that both elements $X(t)$ and $\tilde{X}(t)$ are random variables for all $t \in T$.

The generalizations of relations (1) and (2) for Banach ideal space are given by

$$\lim_{n\to\infty} \left\| \frac{Z_n}{a_n} - \sigma X \right\| = 0\ almost\ surely \tag{4}$$

$$\lim_{n\to\infty} \|Z_n - a_n \sigma X\| = 0\ almost\ surely \tag{5}$$

Relations (4) and (5) can also be considered with respect to the so-called order convergence.

Recall that a sequence of elements (x_n) of a Banach lattice B is said to o-convergence to an element x (we write in this case $x = o - \lim_{n\to\infty} x_n$; se [5,6]) if there is a numerical sequence (v_n) such that $|x - x_n| < v_n$ and $v_n \downarrow 0$. The latter notation means that $v_1 \ge v_2 \ge \cdots$ and $\inf_{n\ge1} v_n = 0$.

A Banach lattice B is called q-concave, $1 \le q < \infty$, if

$$\left(\sum_{i-1}^{n} \|x_i\|^q\right)^{1/q} \le D_{(q)} \left\| \left(\sum_{i-1}^{n} |x_i|^q\right)^{1/q} \right\|$$

for some constant $D_{(q)} = D_{(q)}(B)$.

Put $\tau(F) = \sup(x: F(x) < 1)$,

$$\varphi(y) = inf\left(x \ge 1: \frac{1}{1-F(x)} \ge y\right),$$

$$a_n = \varphi(n) = inf\left(x \geq 1: F(x) \geq 1 - \frac{1}{n}\right). \tag{6}$$

In what follows we assume that $F(x)$ is continuous increasing function and that $\tau(F) = \infty$.

Theorem 1. *Let X be a random element assuming values in a q-concave Banach ideal space B, $1 \leq q < \infty$, for which representation (3) holds. Assume that the function $\varphi(y)$ slowly varies at infinity and that a_n is defined by equality (6).*

We further assume that there exists a number t_0 such that

$$\int_1^{\infty} \frac{1-F(x)}{x\,[1-F(x/t_0)]} dx < \infty \tag{7}$$

and that, for all $t > 1$,

$$\int_1^{\infty} \frac{dF(x)}{1-F(x/t)} dx < \infty. \tag{8}$$

Then

$$o - lim_{n\to\infty} \frac{Z_n}{a_n} = \sigma X \; almost\, surely \tag{9}$$

and

$$E\left\| sup_{n\geq 1} \frac{(Z_n)_+}{a_n} \right\|^q < \infty, \tag{10}$$

where $x_+ = max(x, 0)$ and $x_- = max(-x, 0)$.

Corollary 1. *Relation (4) holds if all the assumptions of Theorem 1 are satisfied.*

Remark 1. *Relation (4) is proved in [4] under the assumption that, for all $t > 1$,*

$$\int_{-\infty}^{+\infty} \frac{|x|^{\,q} dF(x)}{1-F(x/t)} < \infty. \tag{11}$$

Theorem 2. *Let X be a random element assuming values in a q-concave Banach ideal space B, $1 \leq q < \infty$. Assume that representation (3) holds for X and that the sequences a_n is defined by equality (6). We further assume that there exists x_0 such that*

$$\int_{x_0}^{+\infty} \frac{x^q \, dF(x)}{1-F\left((x^q - x_0^q)^{1/q}\right)} < \infty \tag{12}$$

and that, for all $\varepsilon > 0$,

$$\int_1^{+\infty} \frac{dF(x)}{1-F(x-\varepsilon)} < \infty. \tag{13}$$

Then

$o - lim_{n\to\infty}(Z_n - a_n\sigma X) = 0 \; almost \; surely.$ (13)

Corollary 2. *Asymptotic relation (5) holds under the assumptions of Theorem 2.*

Corollary 3. *If assumption (12) in Theorem 2 is changed for the assumption that*

$$\lim_{x\to\infty} \frac{1-F(x+\varepsilon)}{1-F(x)} = 0$$

for all $\varepsilon > 0$ *and if*

$$\int_{-\infty}^{+\infty} |x|^q dF(x) < \infty.$$

then the sequence Z_n *is stable in probability, that is*

$$||Z_n - a_n\sigma X|| \to^P 0.$$

Theorem 1 and 2 are proved in [7].

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